Magnetic Force and Currents

Lecture 25

Physics 321
Electrodynamics I

November 6th, 2006

Motivation

Today we begin to build the theory of magnetostatics. This is the magnetic analogue of the electrostatic theory we have been focusing on since the start of the semester. In the electrostatic case, the distinction between the field $\vec{E}$ and the force on a particle with charge Q: $\vec{F} = Q\vec{E}$ is trivial, and we used our familiarity with point sources for E&M (and Newtonian gravity, for that matter) to dispense with the details of particle motion.

On the magnetic side, the force generated by the magnetic field is quite distinct, both in direction and magnitude, from the field itself, so we will begin by studying the motion of particles, then connect the generation of a magnetic field to its sources, in this case (steady) currents.

Lorentz Force Law

Suppose we have a vector field $\vec{B}(\vec{r})$, which governs particle motion via $\vec{F} = Q\vec{v} \times \vec{B}$. I have not motivated this force, nor told you what $\vec{B}$ is, but we can still explore the trajectories of particles. The force law depends on the particle's charge Q, as with electrostatics, but also on its velocity $\vec{v}$ – this is new, and since we have always been working with stationary charges, it is no surprise that we have not encountered a force like this in electrostatics.

What can we say about a force with this form? Well, it's zero when $\vec{v} = 0$ (or there is no charge), so it is a force that acts only on moving charged particles. In addition, the force is perpendicular to both the instantaneous direction of motion of the particle ($\vec{v}$) and the field $\vec{B}(\vec{r})$, which means that the work done by this type of force must be zero. The units of the field $\vec{B}$ must be $Ns/(mC) = kg/(Cs)$ by dimensional analysis.

Let's take the simplest example, a constant field: $\vec{B} = B_0\hat{z}$. If we think of a particle that starts on the $\hat{x}$ axis at $R\hat{x}$ with initial velocity $\vec{v}(0) = -v_0\hat{y}$, then the equations of motion with initial conditions are:

$$\begin{aligned} m\ddot{x} &= Q\dot{y}B_z \quad m\ddot{y} = -Q\dot{x}B_z \quad m\ddot{z} = 0 \\ \vec{r}(0) &= R\hat{x} \\ \vec{v}(0) &= -v_0\hat{y} \end{aligned} \tag{1}$$

and the configuration is shown in Figure 1.

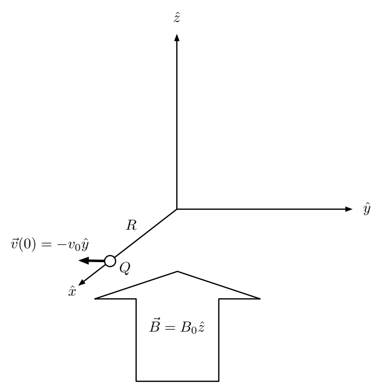

Figure 1: A charge q initially on the $\hat{x}$ axis with velocity in the negative $\hat{y}$ direction. The magnetic field is constant and points "up".

From the equation for z, we have $z(t) = 0$ (in general, $z(t) = \alpha t + \beta$, of course, but our initial velocity and position fix (α, β)). So the motion occurs entirely in the plane. In addition, think of the initial force direction $\vec{v}(0) \times \vec{B}$ – here pointing towards the origin, this suggests we move to a coordinate system in which we can easily describe a vector pointing towards the origin (make it one of the basis vectors): cylindrical. Finally, suppose we look for solutions in which the distance from the origin doesn't change? We expect circular orbits, so let $x(t) = R\cos\phi(t)$, $y(t) = R\sin\phi(t)$, then the equations of motion can be easily transformed:

$$\begin{aligned} \ddot{\phi} &= -\dot{\phi}\left(\frac{B_0Q}{m} + \dot{\phi}\right)\cot\phi \\ \ddot{\phi} &= \dot{\phi}\left(\frac{B_0Q}{m} + \dot{\phi}\right)\tan\phi, \end{aligned} \tag{2}$$

where the top equation corresponds to $\ddot{x}$, the bottom to $\ddot{y}$. The only way to satisfy both of these equations is to have

$$\dot{\phi} = -\frac{B_0 Q}{m} \longrightarrow \phi(t) = -\frac{B_0 Q}{m} t. \tag{3}$$

The particle is in a circular orbit with

$$x(t) = R\cos(\omega t) \qquad y(t) = -R\sin(\omega t) \qquad \omega = \frac{B_0 Q}{m}, \tag{4}$$

the frequency $\omega = \frac{B_0 Q}{m}$ is called the cyclotron frequency. If we have chosen a particular R, then we see from the constant velocity $v^2 = R^2\omega^2$ that the initial velocity $v_0 = R\omega$ is required to get this circular solution. Notice that $v = \omega R$ is just the usual relation between linear velocity and angular velocity at radius R.

Of course, we didn't need to be so fancy – we can obtain all of the above via centripetal acceleration:

$$\frac{mv^2}{R} = QvB_0 \longrightarrow v = \frac{B_0 Q}{m} R = \omega R. \tag{5}$$

Evidently, a constant $\vec{B}$-field supports circular solutions. Again, it is clear in this constant case that this particular force does no work, since for circular motion, the particle moves in the $\hat{\phi}$ direction while the force is in the $-\hat{s}$ direction, these are perpendicular, so that $\vec{v} \cdot \vec{B} = 0$.

$\vec{B}$ is the magnetic field, and it acts on *moving* charged particles according to $\vec{F} = Q\vec{v} \times \vec{B}$. Just as $\vec{E}$, the electric field (acts on charged particles) is sourced by charged particles, the magnetic field $\vec{B}$ will also be sourced by *moving* charged particles.

Combined Force Law

We can now consider the motion of a particle in both $\vec{E}$ and $\vec{B}$ fields – again, we'll keep things constant since we don't really know how to build a magnetic field yet. The force law for the combination (and the combination is typically called the "Lorentz Force Law", although I use it to refer to either portion) acting on charged, moving particles in electric and magnetic fields is

$$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{6}$$

Think of the situation shown in Figure 2, a particle starts at the origin at rest. Because the charge is at rest, there is initially no magnetic force on it, and the electric field accelerates it up the $\hat{z}$ axis – but at that point, the particle has a non-zero velocity in the $\hat{z}$ direction, and $Q\vec{v} \times \vec{B} \sim \hat{y}$, so the charge moves to the right. The larger the velocity, the greater the magnetic force, and it is always directed perpendicular to the motion, so the particle curves around at some radius and heads back down to the $\hat{y}$ axis, against the electric force, so it decelerates until it is finally at rest on the $\hat{y}$ axis and the whole process begins again. A particular version of this so-called "cycloid" trajectory is shown in Figure 2.

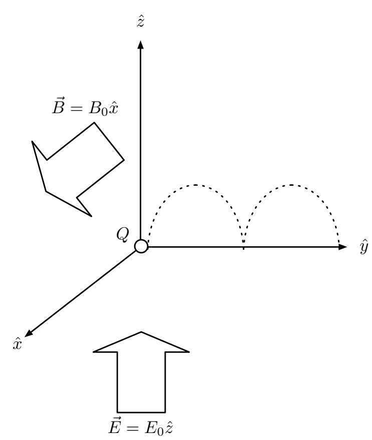

Figure 2: A particle with charge Q starts from the origin at rest in the presence of crossed electric and magnetic fields, $\vec{E} = E_0\hat{z}$, $\vec{B} = B_0\hat{x}$. The cycloid trajectory of the particle's path is shown.

That's the story, anyway, and we can solve for the motion exactly to demonstrate that this is in fact a solution. The frequency of the motion can again be described in terms of $\omega = \frac{QB}{m}$, the cyclotron frequency.

Current

We see that the magnetic field acts on moving charges – that's the experimental observation. It is also "sourced" by moving charges, so that moving charges both react to and generate $\vec{B}$. In order to study the generation of $\vec{B}$ (and develop the Maxwell equations relating $\vec{B}$ to sources), we must have a way to describe continuous distributions of moving charge. As with electrostatics, we will make some simplifying assumptions – most important at this point is that the charge motion should be "steady". A single point charge moving through space is not "steady". If we sit still, the charge comes along and then goes away. So our simplest example of charge motion doesn't fit the bill. Unfortunately, I haven't really defined "steady", we will do that now.

Take an infinite wire of line charge with constant λ, charge per unit length. Suppose we pull it to the right with constant velocity. The charges making up the line are now moving, and we can define the "current" carried in the wire. Current is the charge per unit time passing an observer. If we are sitting somewhere watching the wire move, then in a time dt, the charge that passes us is $\lambda dx = \lambda v dt$. We would call $I = \lambda v$ the current. This is a "steady" current, the amount of charge that passes by us is constant in time, always the same amount.

Current, unlike charge, has a direction. If the wire is moving to the right, and $\lambda > 0$, we say $I > 0$, that's just a convention. If the wire was moving to the left, and $\lambda > 0$, then $I < 0$. But of course, we could also have $I > 0$ for negative charge moving to the left. The product $I = \lambda v$ sets the sign, but physically, in real wires, it is typically electrons that do the moving.

What if the wire isn't straight? We can imagine having a hose of current, with charge flowing like water down the hose. Assuming the same amount of charge flows past us in a given time, this is still a steady current. We can describe the direction of the flow by expanding the notion of sign for I into a full vector – this is implied by the v in $I = \lambda v$ – velocity of a charge is a vector, so to is current: $\vec{I} = \lambda \vec{v}$.

So we can now describe the magnetic force on a segment of the wire with length $d\ell$:

$$d\vec{F} = (dq\vec{v}) \times \vec{B} = (d\ell\lambda\vec{v}) \times \vec{B} = d\ell(\vec{I} \times \vec{B}). \tag{7}$$

since the charge is confined to the wire, we know that $\vec{I} \parallel d\vec{\ell}$, and we can just as easily write

$$d\vec{F} = I(d\vec{\ell} \times \vec{B}). \tag{8}$$

We can integrate the above to get the force on a larger segment:

$$\vec{F} = \int_{\vec{a}}^{\vec{b}} d\vec{F} = \int_{\vec{a}}^{\vec{b}} I(d\vec{\ell} \times \vec{B}) \tag{9}$$

from some point $\vec{a}$ to $\vec{b}$ along the wire.

Example

Take a bent wire segment – suppose it carries a constant current I along it and the top portion is in a constant magnetic field pointing into the page as shown in Figure 3. Obviously, this wire is not an infinite wire being pulled to the right or left, there is some other agency keeping the flow of charge going (a battery, for example).

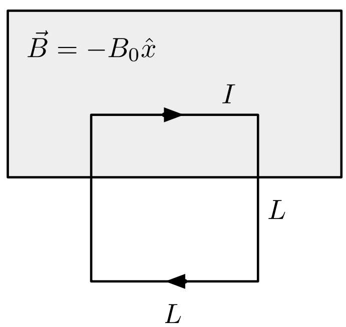

Figure 3: A loop of current-carrying wire with constant I in a constant magnetic field pointing into the page.

Along the top segment of the wire, the portion in the field, we have a force:

$$\vec{F} = \int_0^L I(d\vec{\ell} \times \vec{B}) = I \int_0^L dy B_0(\hat{y} \times (-\hat{x})) = ILB_0\hat{z}, \tag{10}$$

a force pointing upwards with magnitude ILB_0. On the bottom segment, there is no force since there is no magnetic field down there. Along the sides, the force will cancel (on the right, the force points to the right, and on the left, it points to the left). The net upward force will cause the wire to move, which changes the direction of the current. We will return to this later on.

Surface and Volume Current

We have $\vec{I} = \lambda\vec{v}$ as our model for a "line current", but we can also take a sheet of charge and move it. This gives us our model for "surface current". The charge passing by an observer is then $\sigma w dx$ for w, the width of the sheet. We define $\vec{K} = \sigma\vec{v}$ to be the surface current density. The direction of $\vec{K}$ is the direction of the sheet (via $\vec{v}$), the current is $\sigma w v$, with w perpendicular to the motion, so that the magnitude of $\vec{K}$ is the current per unit length perpendicular to the flow.

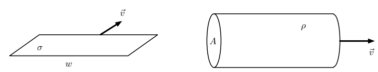

Figure 4: A moving sheet of charge with width w (on the left), and a moving cylinder with uniform ρ (on the right).

Volume charges follow the same pattern (see Figure 4) – our model is a uniformly charged cylinder moving to the right – then the charge passing by an observer is $\rho A dx = (\rho v dt) A$ and $\vec{J} = \rho\vec{v}$ is the volume current density. Again, the direction is parallel to $\vec{v}$, and the magnitude is the current per unit area perpendicular to the flow. Volume current is the most general, and as with ρ, volume charge, we can write the surface current and line current as degenerate forms of $\vec{J}$. It is also easiest to describe total current with $\vec{J}$. Given an arbitrary surface, we can pick out the component of area perpendicular to $\vec{J}$ via $\vec{J} \cdot d\vec{a}$, so the total current through the surface is

$$I = \int_{\mathcal{S}} \vec{J} \cdot d\vec{a}. \tag{11}$$